LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

B.Sc. DEGREE EXAMINATION – MATHEMATICS

FIFTH SEMESTER – APRIL 2010

MT 5508/MT 5502 - LINEAR ALGEBRA

Date & Time: 3/05/2010 / 1:00 - 4:00 Dept. No. Max. : 100 Marks

PART – A (10 X 2 = 20)

Answer ALL questions.

1. Illustrate by an example that union of two subspaces of a vector space need not a subspace.
2. Define linearly dependent and independent vectors in a vector space.
3. When do you say that two vector spaces are isomorphic?
4. Define rank and nullity of a homomorphism.
5. If x is orthogonal to y in an inner product space, show that $\| x + y \|^2 = \| x \|^2 + \| y \|^2$.
6. Define eigenvalues and eigenvectors of a linear transformation.
7. Show that $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is unitary.
8. If A and B are Hermitian matrices, show that $AB + BA$ is Hermitian.
9. What is the characteristic polynomial of the matrix $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$.
10. Let V be an inner product space. Show that for $T \in A(V)$, $T^{**} = T$.

PART – B (5 X 8 = 40)

Answer any FIVE questions.

11. Show that a non-empty subset W of a vector space V over a field F is a subspace of V if and only if $aw_1 + bw_2 \in W$ for all $a, b \in F$, $w_1, w_2 \in W$.
12. If $w_1, w_2, \ldots, w_n \in V$, a vector space, are linearly independent, and if $v \in V$ is not in their linear span, show that $\{v_1, v_2, \ldots, v_n, v\}$ are linearly independent.
13. If V is a vector space of finite dimension and W is a subspace of V, show that

$$\dim V/W = \dim V - \dim W.$$

14. Check whether the following vectors are linearly independent {(1, -1, 0), (1, 3, -1), (5, 3, -2)}
15. State and prove Schwarz inequality.
16. Let $V = \mathcal{R}^3$ and $T \in A(V)$ be defined by $T(a_1, a_2, a_3) = (3a_1 + a_3, -2a_1 + a_2, -a_1 + 2a_2 + 4a_3)$. What is the matrix relative to the basis{ (1, 0, 1), (-1, 2, 1), (2, 1, 1) }.

17. Find the rank of the matrix $\begin{pmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{pmatrix}$.

18. If $< T(v). T(v) > = < v, v >$ for all $v \in V$ an inner product space, show that T is unitary.

SECTION – C (2 × 20 = 40)

Answer any TWO questions.

19. (a) If W_1 and W_2 are subspaces of a finite dimensional vector space V, prove that $dim\ (W_1 + W_2) = dim\ W_1 + dim\ W_2 - dim\ (W_1 \cap W_2)$.

(b) If A and B are subspaces of a vector space V onto a field F, show that $\frac{A+B}{B} \cong \frac{A}{A\cap B}$

20. (a) If W is a subspace of a finite dimensional vector space V, show that $V = W \oplus W^{\perp}$

(b) Prove that $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial for T is not zero.

21. (a) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct eigenvalues of $T \in A(V)$ and if $v_1, v_2, \ldots, v_n$ are eigen vectors of T belonging to $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively, show that $v_1, v_2, \ldots, v_n$ are linearly independent.

(b) Show that for an $m \times n$ matrix A over a field F, the row and the column rank are the same

22. (a) Check the consistency of following set of equations. If it is consistent solve it.

$$x_1 + 2x_2 + 2x_3 = 5, \quad x_1 - 3x_2 + 2x_3 = -5, \quad 2x_1 - x_2 + x_3 = -3.$$

(b) Verify Cayley Hamilton theorem for the following matrix and find A^{-1}.

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

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